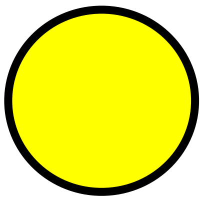
2. SURFACES

§2.1. Disks, Spheres and Cylinders

Topology is a vast area of study and we'll barely scratch the surface. In fact, we'll be studying surfaces.

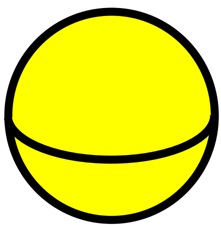
We leave aside the question "what is a surface?" for a later chapter. Here we meet several familiar, and a couple of unfamiliar, examples of surfaces. We begin with the disk, sphere, and cylinder.

A **disk** is the set $\{(x, y) \mid x^2 + y^2 \le r^2\}$, a circle (interior plus boundary) with the usual topology of $\mathbb{R}^2$.



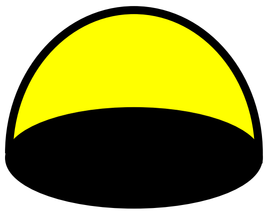
The value of r is the radius of the disk but its actual value, so long as it's positive, is irrelevant to topology.

A **sphere** is the set $\{(x, y, z) \mid x^2 + y^2 + z^2 = r^2\}$ with the usual topology of $\mathbb{R}^3$.

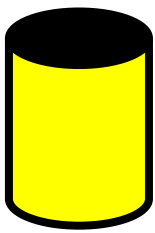


This is the hollow sphere – just the surface. The value of r is the radius of the sphere but its actual value, so long as it's positive, is irrelevant to topology.

A **hemisphere** is the set:
$\{(x, y, z) \mid x^2 + y^2 + z^2 = r^2 \text{ and } z \geq 0\}$ for some $r > 0$.
It has the usual topology of $\mathbb{R}^3$.

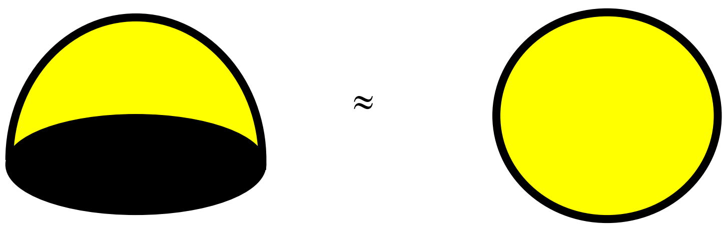


A **cylinder** is the set $\{(x, y, z) \mid x^2 + y^2 = r^2, 0 \leq z \leq h\}$ with the usual topology of $\mathbb{R}^3$.

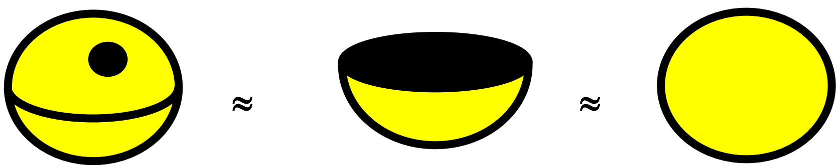


The values of r and h are the radius and height of the cylinder but their actual values, so long as they're positive, are irrelevant to topology.
NOTE: When we talk of a cylinder in topology we refer to the version that's open at both ends.

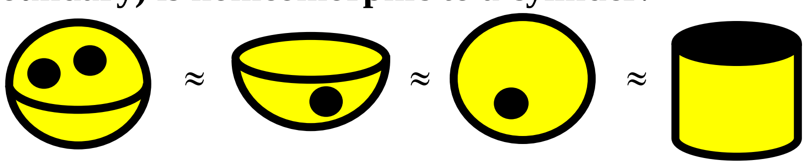
Example 1: A hemisphere is homeomorphic to a disk.
A level 1 explanation would be to imagine the hemisphere being flattened out. A better explanation is one at level 2. Place the hemisphere on a plane and project points on the hemisphere vertically to the plane. The homeomorphism can be easily described as $(x, y, z) \to (x, y, 0)$ and it would not be difficult to prove that it is indeed a homeomorphism and so provide a level 3 proof.



Example 2: A sphere with one hole (including the boundary of the hole) is homeomorphic to a disk.
The hole can be enlarged until it becomes an equator. Then proceed as in example 1.



Example 3: A sphere with two holes (including the boundary) is homeomorphic to a cylinder.



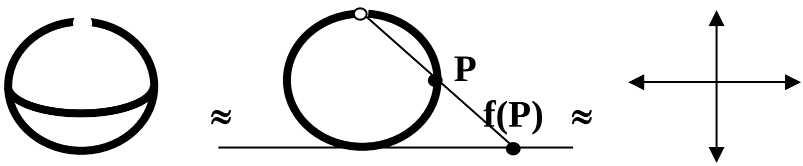
Example 4: The surface of a cone, minus its base, is homeomorphic to a disk.

Again you can place the cone on a plane and project vertically.

Example 5: A punctured sphere (a sphere with a single point removed) is homeomorphic to the entire plane.

A level 1 explanation would be to imagine the area surrounding the missing point being stretched to infinity. Difficult to imagine? I think so! Such a level 1 explanation is quite unsatisfactory.

But all we have to do is to make the missing point the North Pole and place the sphere so that it rests on a plane with just the South Pole touching the plane. Now project from the place where the North Pole would have been, through the sphere and onto the plane. This projection is a homeomorphism and its image is the entire plane.



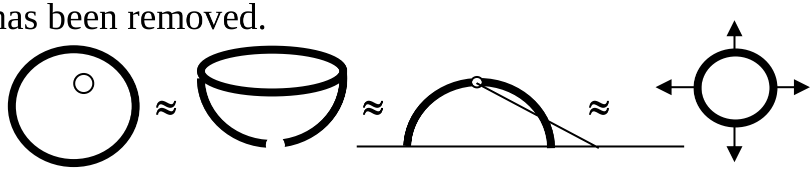
A punctured sphere (single point removed) is homeomorphic to the plane.

[In an informal sense we would certainly call the entire plane a surface. However when we come to define surfaces precisely we'll restrict the term 'surface' to what are generally called **compact surfaces** and the whole plane is not one of those. Never mind what 'compact' means at this stage.]

Example 6: A punctured disk (a disk with one internal point removed) is homeomorphic to the plane with an open disk removed.

(This is also not a surface according to the definition that we'll be giving later. This is because we'll confine our use of the word 'surface' to **closed** spaces. These are spaces where, if you have a sequence of points converging to a limit, the limit is also in the space.)

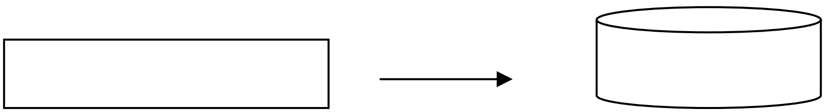
Clearly such a punctured disk is homeomorphic to a punctured hemisphere (with one point removed). Regarding the hemisphere as the Northern Hemisphere, including the equator but excluding the North Pole, we can place this on a plane and then project from the North Pole, through the hemisphere onto the plane. The image of this projection (which clearly is a homeomorphism) is the entire plane with a circular boundary whose interior has been removed.



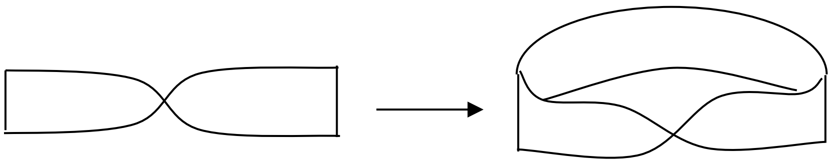
The punctured disk is homeomorphic to the plane with an open disk removed.

§2.2. Möbius Bands

Take a strip of paper, bring the shorter ends directly together and join them and you'll have produced a cylinder. But if you make a half-twist (180º rotation) to one end before joining the ends you'll have created a **Möbius Band**.



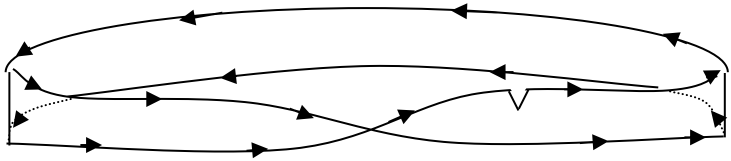
strip(rectangle) **cylinder**



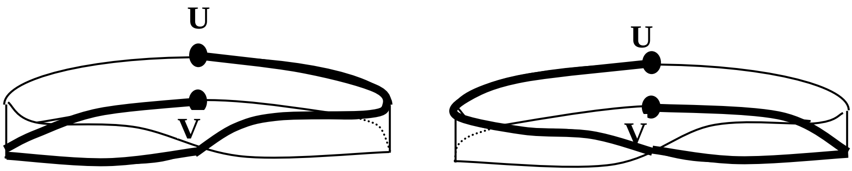
strip with a half twist **Möbius Band**

Before you read further I suggest you make yourself an actual model. The Möbius Band is very much like a cylinder, but topologically they're quite different. For a start a cylinder (with no top or bottom, just the curved side) has two separate boundaries. A Möbius Band only has one.

To see this, make a small V-shaped cut in the edge of your paper model. Now trace around the edge until you return to this mark. Notice that you have traversed the entire boundary. Half-way through you come to the part of the boundary opposite the V-mark but you don't return to it until you've gone around once more.



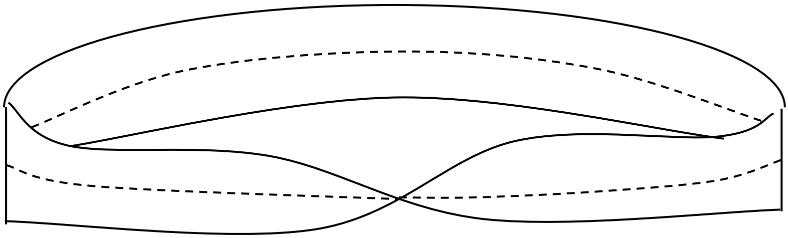
Perhaps a Möbius Band is homeomorphic to a disk. That also has one boundary. But there are topological differences. For a start a disk is simply connected. Any two paths joining any two points are homotopic – one can be continuously deformed into the other. Not so for the Möbius Band. Call the V-cut the point V and name the point opposite it U.



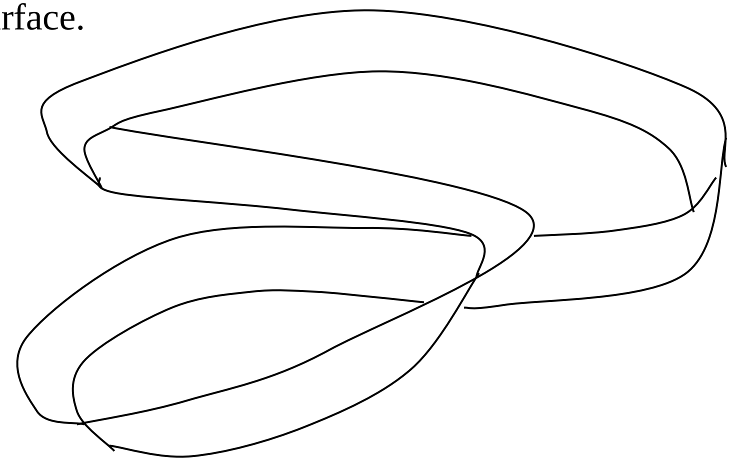
These two diagrams represent two paths from U to V which can't be continuously deformed, one into the other. In other words they aren't homotopic.

Of course, you may not be fully convinced. If so, then that's a good sign. Just because you can't think how to do something (in this case deform one path into the other) doesn't mean it can't be done. It may be that there is some very clever way of doing it that you haven't thought of.

Another difference, and one that's easier to see, is the following. It's possible to remove a closed curve from a Möbius Band in such a way that it remains in one piece. Try it! Just cut along the middle of the Möbius Band. It falls into one larger band, a little twisted, but still in one piece. It's clear that there's no way of cutting along any closed curve on a cylinder without separating it into two pieces.



Cutting a Möbius Band along the centre produces a new surface.



Now look carefully at this new surface. It looks like another Möbius Band. But if you look carefully you'll discover that it has two edges. So it can't be a Möbius Band. Is it a cylinder perhaps? But it looks too twisted to be a cylinder.

In fact this surface *is* homeomorphic to a cylinder. But how can that be? It’s twisted and no amount of deforming it will change it into an untwisted cylinder. That’s true, but an operation that involves cutting, identifying the edges, continuously deforming and then gluing the edges back as they were, is a homeomorphism and we could carry out such an operation to untwist the above surface to obtain a cylinder. The stronger relation that doesn’t permit this, but merely allows continuous deformations, is called ‘homotopy’. We defined it precisely for paths, but it would be too tedious to define it here in general. Suffice to say that the above twisted strip is homeomorphic to a cylinder, but not homotopic to it while the Möbius Band is not even homeomorphic to a cylinder, let alone homotopic.

Cutting is not a homeomorphism, that’s true, and nor is joining. But cutting followed by re-joining is a homeomorphism *provided that the edges of the cut are joined as they were before the cut*. Points that were close together, on opposite sides of the cut, will again be close together.

Cutting and joining are inverse processes. While joining is continuous, cutting is not, so neither of them is a homeomorphism. But cutting and then re-joining, so that points near each other on opposite sides of the cut are brought back close together after the re-joining, *is* indeed a homeomorphism. When a surgeon cuts you open he stitches the two edges back as they were. You might have lost an appendix in the process but, topologically, your surface hasn't changed!

Surgery is the word topologists use to describe the process of cutting and restitching in order to change the appearance of a space without changing it topologically. So you've carried out your first piece of topological surgery.

§2.3. Toruses

The correct mathematical name for a doughnut shape is 'torus'. We can mean either a solid torus or just the surface of a torus. Here, since we're only discussing surface topology, we mean the *surface* of a torus.

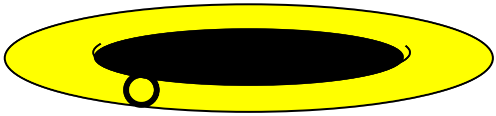
If the circumference of a circle is rotated through 360^{o} about a line that lies in the same plane as the circle but does not intersect it, the circle that the surface traced out is a torus. We can define a torus parametrically.

A **torus** is the set:

$$\{((a + b\cos\varphi)\cos\theta, (a + b\cos\varphi)\sin\theta, b\sin\varphi)\}$$

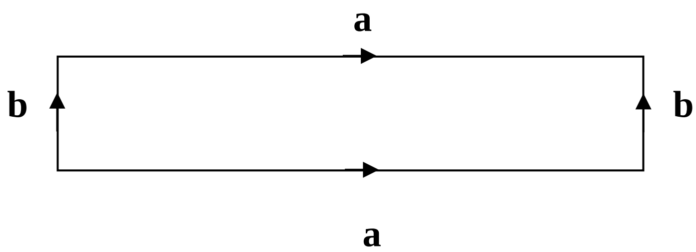
$$\text{for } 0 \leq \theta, \varphi < 2\pi$$

for fixed real numbers a, b with $a > b$. (This torus is the locus of a circle of radius b that is rotated around an axis at a distance a from its centre.)

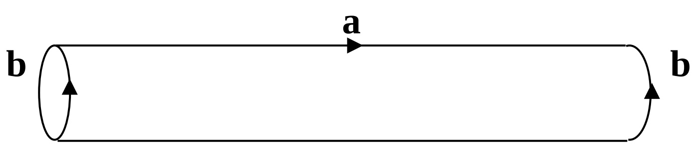


A plastic tube is topologically a cylinder. If you bring the two ends together and glue them, you have a torus. On the other hand if you slit the cylindrical tube along its length you get a long, thin rectangle, topologically a disk.

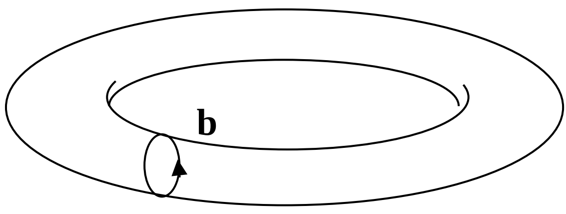
So, if you take a long, thin, sufficiently flexible rectangle you can join it up along the longer sides to form a cylinder and then join the ends of the cylinder to form a torus. We could give instructions for doing this on a sheet of paper as follows:



(1) Cut out along the four edges.
(2) Join the two edges marked **a** (make sure the arrows point in the same direction.).



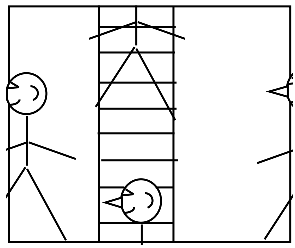
(3) Now join the two edges marked **b** (again make sure the arrows match up).



Unfortunately paper is the wrong material for doing this sort of thing so I don't want you to actually follow the recipe. I just want to point out the way labels can be used to describe which edges are to be brought together.

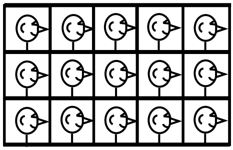
This is what we call **identifying edges**. We don't actually have to bring the identified edges together – merely labelling them appropriately is sufficient.

So a torus is topologically the same as a rectangle with opposite pairs of edges identified. This means that if you lived on such a rectangle and walked towards an edge you'd continue from the corresponding point on the opposite edge. Sound familiar? Of course! It's 'wrap-around' – this time top and bottom wrap-around as well as left-right wrap-around. Some computer games use both directions. A character on the screen climbs a ladder and disappears off the top of the screen only to reappear at the bottom, still climbing.



Here's an even stranger way of thinking about the torus. Imagine that you're a microscopic insect crawling around a large sheet of postage stamps. Suppose you don't know about stamps or the fact that they're often produced in large sheets of identical stamps in rows and columns. But you can detect the patterns of ink on the paper and, what's more, you're a topologically trained insect.

As you move up a particular stamp you cross onto another stamp in the next row up. Since it's identical to the one you've just come from you assume, quite naturally, that you've returned to the same one.



I once heard of a maze, involving several interconnecting rooms. While most of the rooms were distinctively different a few were identical to others in pairs. So coming to a room identical to one you'd been in you took it to be the same one, and as you built up a picture in your mind of the overall layout you got very, very confused!

To a microscopic insect, wandering about a sheet of postage stamps, it would appear that it was on a topological torus (unless it ever wandered far enough that it came to the edge of the whole sheet).

A torus has no boundaries, so it can't be homeomorphic to a disk, or a cylinder, or a Möbius Band. But is it perhaps homeomorphic to a sphere? The answer is "no" because a sphere is simply connected while a torus is not.

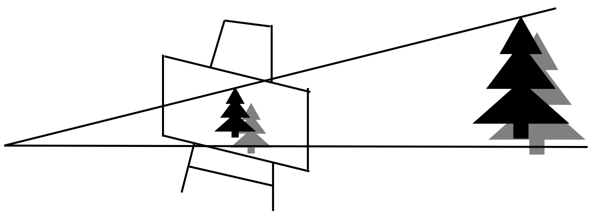
§2.4. The Projective Plane

The next surface is one you've never seen. And you never will. The projective plane can't exist in three-dimensional space. So unless you've ever slipped into a 4-dimensional world you'll never have seen one.

But aren't we talking about surfaces, and aren't surfaces just 2-dimensional subsets of $\mathbb{R}^3$? We're sticking to 2 dimensions certainly but we don't limit ourselves to 3 dimensions for these surfaces to live in.

We can, in fact, provide a model for the projective plane in $\mathbb{R}^3$, but it involves a bit of trickery. The projective plane doesn't exist, as a set of *points* in $\mathbb{R}^3$, but it does exist in $\mathbb{R}^3$ as a set of **lines**!

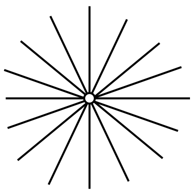
A painter painting a scene doesn't represent points in the scene by points on the canvas, but rather he or she represents *rays*. If two objects form a straight line with the painter's eye they'll correspond to the same point on the canvas. One will obscure, or partly obscure the other.



So rays, from the painter's eye extending out to infinity, are what get painted. But not all rays get painted though because the canvas is only so big. And rays that radiate out through the back of the painter's head don't get into the picture at all. But let's leave the painter behind and consider *all* lines through the origin.

These lines are what are called **projective points** in projective geometry. Can a line be a point? It can if we choose to call it a point. So our 'points' in this topological space are what we'd normally call lines through the origin. We now have to set up the topology on this set by defining neighbourhoods.

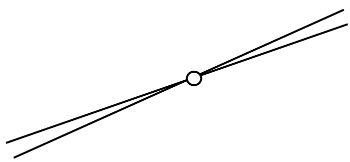
This we do as 'pencils' or cones of lines. If θ is any real number with $0 < \theta < \pi$ we define the neighbourhood of a line L through the origin to be the set of all lines through the origin that make an angle less than θ with L.



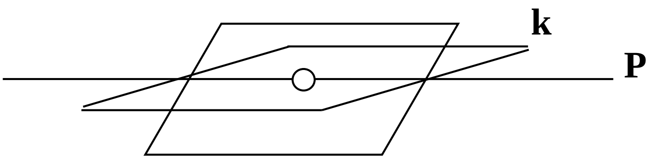
With this picture you have to imagine some of the lines sticking out of the page. If you stare at this picture long enough it will look like that.

So a line through the origin (don't forget to call this a 'projective point') is close to another if the angle between them is very small. The smaller the angle, the closer they are together.

Two close projective points



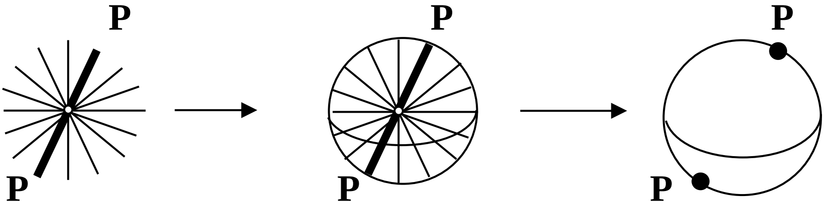
If projective points are lines through the origin what are projective lines? You've guessed it. They're planes through the origin.



Two projective lines h, k intersecting in a projective point P.

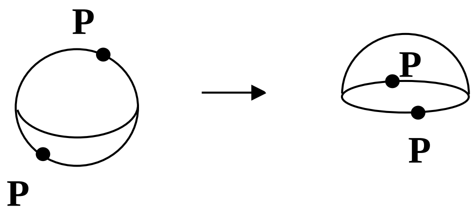
What makes the projective plane interesting from a geometric point of view is that any two projective lines intersect in a projective point. There are no such things as parallel lines in projective geometry. To see this you just have to translate the words. All we're saying really is that any two planes through the origin intersect in a line through the origin – which is true.

But we're not doing projective geometry as such. We're more interested here in the topology of the projective plane. One picture of it is as a sort of porcupine, with lines radiating out from a single point. But we can surround this point by a sphere and each of these lines will cut the sphere in two antipodal points. We can now throw away the lines and just keep the points. Each projective point (line through the origin) is now represented by a pair of antipodal points. And since both of these points represent the same projective point we have to identify pairs of antipodal points.



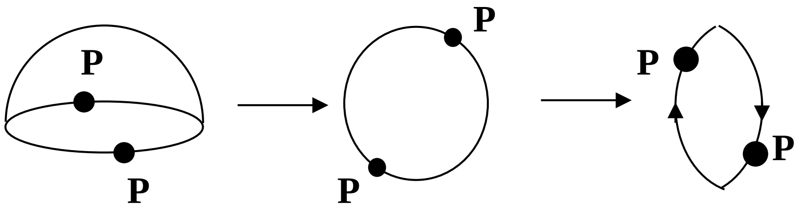
The Projective Plane

The real projective plane is thus homeomorphic to a sphere in which antipodal points are identified. We can go further to obtain an even simpler model. Since each point in the northern hemisphere has its equal double in the south, why not cut off the entire bottom half of this sphere and so obtain a hemisphere. But it’s not an ordinary hemisphere. Points on the equator still have to be identified in pairs.



The Projective Plane

We can now flatten the hemisphere and so obtain a circle with opposite points identified. And now we squash the circle so that it looks more like a flat banana and obtain a 2-sided polygon, with the two sides identified.



The Projective Plane

But it's important to describe in what way the two sides are identified. If we run the arrows the same way we get a figure where the two identified edges can be easily brought together and zipped up to produce the surface of a banana shape. Topologically that's just a sphere.

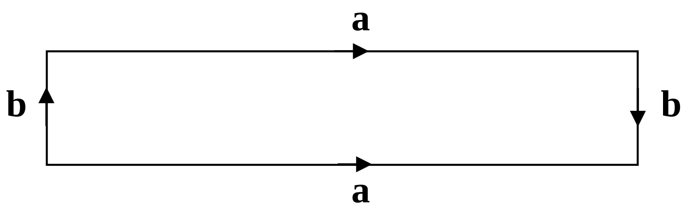
Sphere **Projective Plane**

But with opposite points being identified, the way the arrows have to be matched up is different. For the projective plane there's clearly no way these edges can be joined up so that the arrows match up – not in 3-dimensional space that is. There would be no problem in 4-dimensional space but, since we're not too worried about where these surfaces live, we won't pursue that matter.

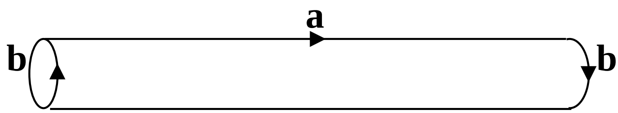
§2.5. Klein Bottles

Just as weird as the Projective Plane is the Klein Bottle. Like the Projective Plane, it doesn't live in ordinary 3-dimensional space. But we can still talk about it.

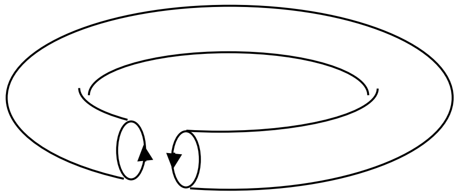
A Klein Bottle is formed by taking a cylinder and joining the two circular ends with the opposite orientation to what would be needed to form a torus. In order for the ends to be brought together, so that the arrows matched up, we'd need to bring one end around and behind the other. The trouble is that we'd have to pass it from the outside of the cylindrical tube into the inside without passing through the walls of the tube. It can't be done – not in $\mathbb{R}^3$.



Notice the subtle difference between this and the model of a cylinder. The orientations of the **b** arrows are opposite. When we join along the **a** arrows we get:

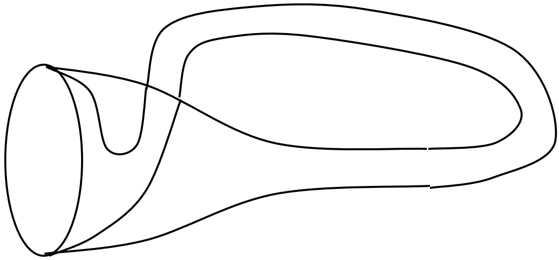


If we now roll this up we discover that the circular ends have the wrong orientation to be joined up as in a cylinder.



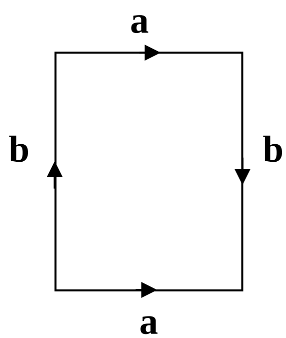
What we have to do is to bring one of the ends over and join up from behind. In three-dimensional Euclidean space the only way we can do this is to pass one end through the surface near the other end. We can do this to approximate the Klein Bottle, but in the true Klein Bottle there is no self-intersection. Needless to say, the Klein Bottle isn't a surface that lives in ordinary three-dimensional space. In other words there's no subset of $\mathbb{R}^3$ (with the usual topology) that is homeomorphic to the Klein Bottle.

I once saw a glass model of a Klein Bottle made by a skilled glassblower. Of course he cheated – he had to. One end passes through the bottle so that it can join up with the other in the twisted way that it's supposed to. You just have to pretend that it's not actually intersecting itself. It's called a 'bottle' because you could try to put something in it. The only trouble is that the inside surface is the same as the outside surface, which could cause problems.



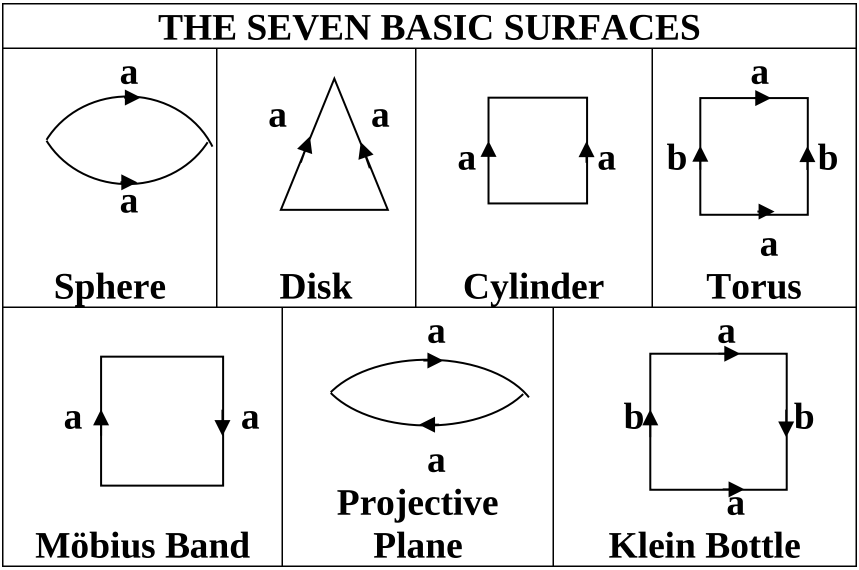
A Euclidean approximation to the Klein Bottle

The simplest way to describe the Klein Bottle is to represent it as a square with opposite edges identified, just as with the torus, but with one pair of identified edges having the opposite orientation.



The Klein Bottle

We've now met the 7 fundamental surfaces. Here they are, together with representations as polygons-with-identified edges:



These are the only surfaces that have special names, but there are infinitely many other surfaces. The surprising thing is that they can all be built up from just three of the above. Disks, projective planes and toruses are the building blocks from which all surfaces can be created.

EXERCISES FOR CHAPTER 2

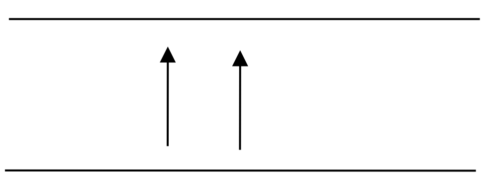
Exercise 1: Take a long strip of paper and give it two half twists before joining the ends. Which of the above seven surfaces is it homeomorphic to. Give reasons.

Exercise 2: Make a Möbius Band out of paper. (Either use sticky tape to join the ends or use staples. If you use staples you need 4 staples across the width so that the pieces stay together.) Now cut along the centre of the strip. Then take the new strip and cut along its centre. Describe what you get at each stage.

Exercise 3:
Find all possible surfaces that can be obtained from a square by identifying some, or all, of its edges in pairs.

SOLUTIONS FOR CHAPTER 2

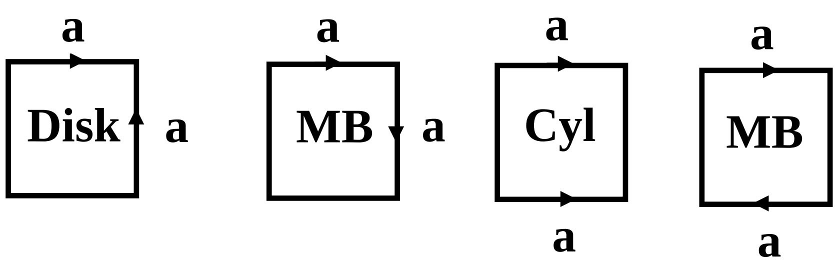
Exercise 1: The strip with two half twists is homeomorphic to a cylinder when the ends are joined. To see this, carry out the following operations. Draw two arrows, side by side, pointing in the same direction at right angles to the strip.



Now cut across the strip between the two arrows. Untwist the strip and rejoin. Note that the arrows will be pointing in the same direction on either side of the join. Since this process is reversible, and points that were close together stay close together, this cut, untwist and rejoin operation is a homeomorphism. But the untwisted strip, with the ends joined, is a cylinder.

Exercise 2:
Cutting the Möbius Band along its centre you get one longer strip, twice the length and half the width. Moreover the strip is twisted, but a careful examination shows that it is homeomorphic to a cylinder. Cutting this again along its centre you get two intertwined pieces, each of which is twisted but is homeomorphic to a cylinder.

Exercise 3:
A polygon-with-identified-edges with 4 sides can have 0, 1 or 2 pairs of identified edges. With no identified edges we get the disk. With one pair of identified edges we can have them adjacent or opposite and in each case the pair can be like or unlike (going in the same direction or opposite directions when you go around the square). This gives the 4 possibilities:



With two pairs of identified edges they can be either adjacent or interleaved and in each case there can be no like pairs, one like pair or two like pairs. This gives 6 possibilities: